DIMENSION-FREE L^p ESTIMATES FOR HIGHER ORDER MAXIMAL RIESZ TRANSFORMS IN TERMS OF THE RIESZ TRANSFORMS

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Let P be a harmonic homogeneous polynomial of degree k. In the talk we will consider higher order Riesz transforms defined as

$$R_P f(x) = \lim_{t\to 0^+} R_P^t f(x), \qquad \text{where} \qquad R_P^t f(x) = \frac{\Gamma\left(\frac{k+d}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{k}{2}\right)} \int_{|y|>t} \frac{P(y)}{|y|^{d+k}} f(x-y)\,dy,$$

and their relation with the corresponding maximal Riesz transforms given by

$$R_P^* f(x) = \sup_{t>0} \left|R_P^t f(x)\right|.$$

In particular we will sketch a proof of the fact that the $L^p(\mathbb{R}^d)$ norm of the vector of maximal Riesz transforms can be controlled in a dimension-free way by the $L^p(\mathbb{R}^d)$ norm of the vector of the corresponding Riesz transforms, namely

Theorem. *Take $p \in (1,\infty)$ and let $k \leqslant d$ be a non-negative integer. Let $\mathcal{P}_k$ be a set of harmonic homogeneous polynomials of degree k. Then there is a constant $A(p,k)$ independent of the dimension d and such that*

$$\left\|\left(\sum_{P\in\mathcal{P}_k} |R_P^* f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} \leqslant A(p,k) \left\|\left(\sum_{P\in\mathcal{P}_k} |R_P f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)},$$

where $f \in L^p(\mathbb{R}^d)$. Moreover, for fixed k we have

$$A(p,k) = O(p^{5/2+k/2}) \quad \text{as } p\to\infty \qquad \text{and} \qquad A(p,k) = O((p-1)^{-5/2-k/2}) \quad \text{as } p \to 1.$$

The research was inspired by the results of Mateu, Orobitg, Pérez and Verdera [3, 4, 5], who proved that for $1 < p < \infty$ there is a constant $C_{p,k,d}$ depending on p, k and d such that

$$\|R_P^* f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,k,d} \|R_P f\|_{L^p(\mathbb{R}^d)}.$$

The proof consists of four parts:

(1) We factorize the operator R_P^t into $R_P^t = M_k^t(R_P)$.
(2) We express the operator M_k^t in terms of Riesz transforms

$$M_k^t f(x) = C(d,k) \int_{SO(d)} \sum_{j\in I} (R_j^t R_j f)_U(x)\,d\mu(U),$$

where T_U is the conjugation of an operator T by $U \in SO(d)$ and I denotes the set of multi-indices $j = (j_1,\dots,j_k)$ with pairwise distinct elements.
(3) We extend the operator $R^t = \sum_{j\in I} R_j^t R_j$ on $\mathbb{R}^d$ to the operator $\widetilde{R}^t$ on $\mathbb{C}^d$ and apply the complex method of rotations of Iwaniec and Martin [1] in order to express $\widetilde{R}^t$ in terms of the complex Hilbert transform.
(4) We deduce the estimates for R^t from the estimates for $\widetilde{R}^t$.

The talk is based on joint work with Błażej Wróbel and Jacek Zienkiewicz [2].

References

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