Matematički institut SANU
Odeljenje za matematiku

Matematički fakultet u Beogradu
Opšti matematički seminar

PLAN RADA ZA OKTOBAR 2012.

Petak, 02.11.2012. 14 časova, sala 2 SANU !!! OBRATITE PAŽNJU NA MESTO !!!
MATEMATIČKI INSTITUT SANU

Ivan Gutman, Prirodno-matematički fakultet, Univerzitet u Kragujevcu

O ENERGIJI GRAFA I O KNJIZI ''GRAPH ENERGY''

Rezime: Neka je G graf sa n čvorova, i neka su λ_1, λ_2, … λ_n, njegove sopstvene vrednosti. Tada je *energija grafa G* definisana kao $\sum_{i=1}^{n} |\lambda_i|$. Ovu definiciju je dao predavač godine 1978, nadajući se da će kolege matematičari uvideti da ova grafovska invarijanta ima niz interesantnih i netrivijalnih osobina. Oni su to zaista uvideli, ali im je za to trebalo nešto više od dvadeset godina. Početkom 21. veka, širom naše planete započinju matematička istraživanja o energiji grafova, koja su do sada rezultovala u nekoliko stotina publikacija. Na predavanju će, osim kraće istorije ove problematike, biti izloženi važniji rezultati i važniji nerešeni problemi o energiji grafova, a prikazaćemo i knjigu ''*Graph Energy*'', Springer, New York, 2012.

Petak 16. novembar 2012, 14h sala 301f
MATEMATICKI INSTITUT SANU

Veljko A. Vujičić: Matematički institut SANU

FOUR-DIMENSIONAL SPACES WITH GEOMETRIC AND KINEMATIC CONSTRAINTS: "Argumentima protiv negativne recenzije"

Petak 30. novembar 2012, 14h sala 301f
MATEMATICKI INSTITUT SANU

Dragan S. Djordjević, Faculty of Sciences and Mathematics, University of Niš

GENERALIZED INVERSES OF OPERATORS ON HILBERT *C-MODULES**

Abstract: Follows at the end of file.

Odeljenje za matematiku Matematičkog instituta SANU
Opšti matematički seminar na Matematičkom fakultetu u Beogradu

Rukovodioci *Odeljenja za matematiku* Matematičkog instituta SANU i *Opšteg matematičkog seminara* na Matematičkom fakultetu u Beogradu, Stevan Pilipović i Siniša Vrećica predlažu zajednički program rada naučnih sastanaka.

Predavanja će se održavati u Matematičkom Institutu SANU (sala 2), petkom sa početkom u 14 časova. *Odeljenje za matematiku* je opšti seminar sa najdužom tradicijom u Institutu.

Svakog meseca, jedno predavanje će biti održano na Matematičkom fakultetu u terminu koji će biti posebno odredjen.

Molimo sve zainteresovane učesnike u radu naučnih sastanaka da posebno obrate pažnju na
vreme održavanja svakog sastanka. Na Matematičkom fakultetu su moguće izmene termina.

Obaveštenje o programu naučnih sastanaka će biti objavljeno na oglasnim tablama MI SANU (Beograd), MF (Beograd), PMF (Novi Sad), PMF (Niš) i PMF (Kragujevac).

Predavanja su namenjena širokom krugu matematičara - i onima koji ne rade u toj oblasti.
POSEBNO SU DOBRODOŠLI POSTDIPLOMCI I STUDENTI STARIJIH GODINA

Odeljenje za matematiku
Matematičkog instituta SANU

Stevan Pilipović

Opšti matematički seminar na
Matematičkom fakultetu u Beogradu,

Siniša Vrećica

Ovo obaveštenje možete naći i na Internetu: **www.mi.sanu.ac.rs**
Ako želite da se obaveštenje o Vašim naučnim skupovima pojavi u Newsletter of EMS (European Mathematical Society) i na Internetu na lokaciji EMS, onda se obratite na
emsvesti@mi.sanu.ac.rs
gde ćete dobiti format obaveštenja.

Generalized inverses of operators on Hilbert C^*-modules

Dragan S. Djordjević

November 22, 2012

Let A be a C^*-algebra and let $\mathcal{M}$ be a right A-module. This means that $(\mathcal{M}, +)$ is an Abelian group, and there exists an exterior multiplication: if $x \in \mathcal{M}$ and $a \in A$, then $x \cdot a \in \mathcal{M}$. This multiplication satisfies the same axioms as the scalar multiplication in vector spaces.

Additionally, if A does not have the unit, we assume that the scalar multiplication of elements in $\mathcal{M}$ exists. If $\lambda \in \mathbb{C}$ and $x \in \mathcal{M}$, then we write equivalently $x\lambda = \lambda x \in \mathcal{M}$. If A has the unit, then the scalar multiplication follows easily from the multiplication by elements of A.

Definition 0.1. Let $\mathcal{M}$ be a module over a C^*-algebra A. Suppose that there exists an A-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A$, satisfying the following:

(1) $\langle x, x \rangle \geq 0$ in A for all $x \in \mathcal{M}$;

(2) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{M}$;

(3) $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in \mathcal{M}$ and all $z \in A$.

Then $\mathcal{M}$ is a Hilbert pre-module over A.

Definition 0.2. If M is a pre-Hilbert module over A, and $\mathcal{M}$ is complete with respect to the norm $\| \cdot \|_{\mathcal{M}}$, then $\mathcal{M}$ is a Hilbert C^*-module over A, or $\mathcal{M}$ is a Hilbert C^* A-module.

Example 0.1. If A is a C^*-algebra, then A is itself a Hilbert module, since the inner product is given by $\langle a, b \rangle = a^* b$ for all $a, b \in A$.

More generally, let J be a right ideal of A. Then J is a Hilbert module over A, if the inner product is given by $\langle a, b \rangle = a^* b$.

Example 0.2. Let $M^{m\times n}$ denotes the set of all complex matrices of the form $m \times n$. Then $A^{m\times n}$ is a right $M^{n\times n}$-module. The norm $\|.\|$ can be defined as $\|A\|_{A^{m\times n}} = \|AA^*\|$.

On the other hand, we can consider $A^{m\times n}$ as a left $A^{m\times m}$-module, and the natural norm is defined as $\|A\|_{A^{m\times n}} = \|A^* A\|$.

We know that both norms are the same!

Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a C^*-algebra A. A mapping $T : \mathcal{M} \to \mathcal{N}$ is called *operator* if T is a bounded $\mathbb{C}$-linear A-homomorphism from $\mathcal{M}$ to $\mathcal{N}$, i.e. T satisfies:

$$T(x+y) = T(x)+T(y),\ T(\lambda x) = \lambda T(x),\ T(xa) = T(x)a, \quad x, y \in \mathcal{M},\ a \in A,\ \lambda \in \mathbb{C},$$

and there exists some $M \geq 0$ such that

$$\|T(x)\|_{\mathcal{M}} \leq M\|x\|_{\mathcal{N}},\ x \in \mathcal{M}.$$

The norm of T is given by

$$\|T\| = \inf\{M \geq 0 : \|T(x)\|_{\mathcal{M}} \leq M\|x\|_{\mathcal{N}},\ \text{for all } x \in \mathcal{M}\}.$$

The set of all operators from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. Particularly, $\mathrm{End}_A(\mathcal{M}) = \mathrm{Hom}_A(\mathcal{M}, \mathcal{M})$.

Lemma 0.1. $\mathrm{End}_A(\mathcal{M})$ *is a Banach algebra.*

We shall see that the question of adjoint operators is not trivial.

Lemma 0.2. *Let $\mathcal{M}$ be a Hilbert A-module, and let $T : \mathcal{M} \to \mathcal{M}$ and $T^* : \mathcal{M} \to \mathcal{M}$ be A-linear mappings such that*

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \quad \text{for all } x, y \in \mathcal{M}.$$

Then $T, T^* \in \mathrm{End}_A(\mathcal{M})$.

Definition 0.3. An operator $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ is adjointable, if there exists and operator $T^* \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that for all $x \in \mathcal{M}$ and all $y \in \mathcal{N}$ the following holds:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

There exists operators that are not adjointable.

The set of all adjointable operators from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}^*_A(\mathcal{M}, \mathcal{N})$. We see that $\mathrm{End}^*_A(\mathcal{M})$ is a C^*-algebra.

Theorem 0.1. *For $T \in \mathrm{End}^*_A(\mathcal{M})$ the following conditions are equivalent:*

(1) *T is a positive element in the C^*-algebra $\mathrm{End}^*_A(\mathcal{M})$;*

(2) *For all $x \in \mathcal{M}$ the element Tx is positive in the C^*-algebra A.*

Theorem 0.2. *Let $T : \mathcal{M} \to \mathcal{N}$ be a linear map. Then the following statements are equivalent:*

(1) *T is bounded and A-homomorphism;*

(2) *There exists a constant $K \geq 0$ such that the inequality $\langle Tx, Tx\rangle \leq K\langle x, x\rangle$ holds in A for all $x \in \mathcal{M}$.*

Lemma 0.3. *Let A be a unital C^*-algebra and let $r : A \to A$ be a linear map such that for some constant $K \geq 0$ the inequality $r(a)^* r(a) \leq K a^* a$ holds for all $a \in A$. Then $r(a) = r(1)a$ for all $a \in A$.*

Example 0.3. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ be the orthogonal decomposition of Hilbert modules. Define $P : \mathcal{M} \to \mathcal{M}$ to be the projection from $\mathcal{M}$ onto $\mathcal{N}$ parallel to $\mathcal{L}$. Then P is bounded, $\|P\| = 1$ and $P^* = P$. Hence, $P \in \mathrm{End}_A^*(\mathcal{M})$.

Theorem 0.3. (Misčenko) *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, and let $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$ such that $R(T)$ is closed in $\mathcal{N}$. Then the following hold:*
(1) *$N(T)$ is a complemented submodule of $\mathcal{M}$ and $N(T)^\perp = R(T^*)$;*
(2) *$R(T)$ is a complemented module of $\mathcal{N}$ and $R(T)^\perp = N(T^*)$;*
(3) *T^* also has a closed range.*

Let $\mathcal{M}, \mathcal{N}$ be Hilbert modules, and let $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$, or $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$.
T is generalized invertible, if there exists some $T_1 \in \mathrm{Hom}(\mathcal{N}, \mathcal{M})$ such that $TT_1T = T$.
We can also require that S satisfies all Penrose eqautions, in order to obtain the Moore-Penrose inverse of T.

Outer inverse with prescribed range and null-module:
Let $T \in \mathrm{Hom}_A^*(\mathcal{M}, \mathcal{N})$, and let K and H be submodules of $\mathcal{M}$ and $\mathcal{N}$, respectively. Find $U \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that the following hold:

$$UTU = U, \;\; R(U) = K, \;\; N(U) = H.$$

If such U exists, then $U = T^{(2)}_{K,H}$.
Equivalent conditions (Xu, Zhang):

$$\mathcal{N} = A(K) \oplus H, \;\; N(T) \cap K = \{0\}, \;\; \mathcal{M} = T^*(H^\perp) \oplus K^\perp), \;\; N(T^*) \cap H^\perp = \{0\}.$$

The notion for the commutators follows: $[U, V] = UV - VU$, for appropriate choice of operators U and V.

Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be Hilbert modules, and let $A \in \mathrm{Hom}^*(\mathcal{N}, \mathcal{L})$ and $B \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ have closed ranges, such that AB also has a closed range. Find necessary and sufficient conditions such that the reverse order law holds:

$$(AB)^\dagger = B^\dagger A^\dagger.$$

A new result follows.

Theorem 0.4. *If $A \in \mathrm{Hom}^*(\mathcal{N}, \mathcal{L})$, $B \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ and $AB \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ have closed ranges, then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $[A^\dagger A, BB^*] = 0$ *and* $[A^* A, BB^\dagger] = 0$;

(3) $R(A^*AB) \subset R(B)$ *and* $R(BB^*A^*) \subset R(A^*)$;

(4) A^*ABB^* *has a commuting Moore-Penrose inverse.*