The Abel Prize Laureates 2015

John Forbes Nash, Jr.

Princeton University, USA

Louis Nirenberg

Courant Institute, New York University, USA

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John F. Nash, Jr. and Louis Nirenberg receive the Abel Prize for 2015

“for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis.”

Citation

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2015 to

John F. Nash, Jr., Princeton University and **Louis Nirenberg**, Courant Institute, New York University

"for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis."

Partial differential equations are used to describe the basic laws of phenomena in physics, chemistry, biology, and other sciences. They are also useful in the analysis of geometric objects, as demonstrated by numerous successes in the past decades.

John Nash and Louis Nirenberg have played a leading role in the development of this theory, by the solution of fundamental problems and the introduction of deep ideas. Their breakthroughs have developed into versatile and robust techniques, which have become essential tools for the study of nonlinear partial differential equations. Their impact can be felt in all branches of the theory, from fundamental existence results to the qualitative study of solutions, both in smooth and non-smooth settings. Their results are also of interest for the numerical analysis of partial differential equations.

Isometric embedding theorems, showing the possibility of realizing an intrinsic geometry as a submanifold of Euclidean space, have motivated some of these developments. Nash's embedding theorems stand among the most original results in geometric analysis of the twentieth century. By proving that any Riemannian geometry can be smoothly realized as a submanifold of Euclidean space, Nash's smooth (C^∞) theorem establishes the equivalence of Riemann's intrinsic point of view with the older extrinsic approach. Nash's non-smooth (C1) embedding theorem, improved by Kuiper, shows the possibility of realizing embeddings that at first seem to be forbidden by geometric invariants such as Gauss curvature; this theorem is at the core of Gromov's whole theory of convex integration, and has also inspired recent spectacular advances in the understanding of the regularity of incompressible fluid flow. Nirenberg, with his fundamental embedding theorems for the sphere S2 in R3, having prescribed Gauss curvature or Riemannian metric, solved the classical problems of Minkowski and Weyl (the latter being also treated, simultaneously, by Pogorelov). These solutions were important, both because the problems were representative of a developing area, and because the methods created were the right ones for further applications.

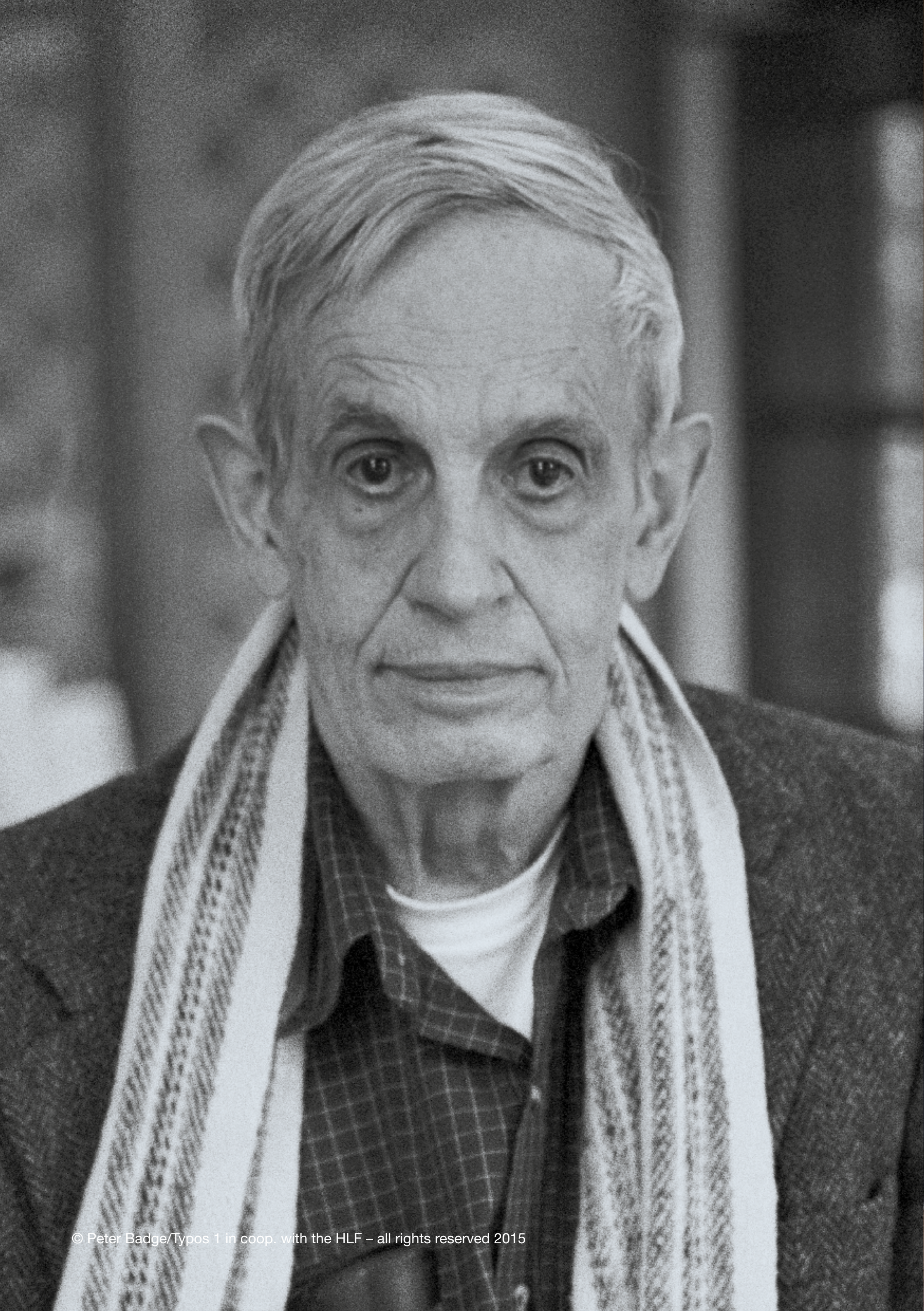
Nash's work on realizing manifolds as real algebraic varieties and the Newlander-Nirenberg theorem on complex structures further illustrate the influence of both laureates in geometry.

Regularity issues are a daily concern in the study of partial differential equations, sometimes for the sake of rigorous proofs and sometimes for the precious qualitative insights that they provide about the solutions. It was a breakthrough in the field when Nash proved, in parallel with De Giorgi, the first Hölder estimates for solutions of linear elliptic equations in general dimensions without any regularity assumption on the coefficients; among other consequences, this provided a solution to Hilbert's 19th problem about the analyticity of minimizers of analytic elliptic integral functionals. A few years after Nash's proof, Nirenberg, together with Agmon and Douglis, established several innovative regularity estimates for solutions of linear elliptic equations with Lp data, which extend the classical Schauder theory and are extremely useful in applications where such integrability conditions on the data are available. These works founded the modern theory of regularity, which has since grown immensely, with applications in analysis, geometry and probability, even in very rough, non-smooth situations.

Symmetry properties also provide essential information about solutions of nonlinear differential equations, both for their qualitative study and for the simplification of numerical computations. One of the most spectacular results in this area was achieved by Nirenberg in collaboration with Gidas and Ni: they showed that each positive solution to a large class of nonlinear elliptic equations will exhibit the same symmetries as those that are present in the equation itself.

Far from being confined to the solutions of the problems for which they were devised, the results proved by Nash and Nirenberg have become very useful tools and have found tremendous applications in further contexts. Among the most popular of these tools are the interpolation inequalities due to Nirenberg, including the Gagliardo-Nirenberg inequalities and the John-Nirenberg inequality. The latter governs how far a function of bounded mean oscillation may deviate from its average, and expresses the unexpected duality of the BMO space with the Hardy space H1. The Nash-De Giorgi-Moser regularity theory and the Nash inequality (first proven by Stein) have become key tools in the study of probabilistic semigroups in all kinds of settings, from Euclidean spaces to smooth manifolds and metric spaces. The Nash-Moser inverse function theorem is a powerful method for solving perturbative nonlinear partial differential equations of all kinds. Though the widespread impact of both Nash and Nirenberg on the modern toolbox of nonlinear partial differential equations cannot be fully covered here, the Kohn-Nirenberg theory of pseudo-differential operators must also be mentioned.

Besides being towering figures, as individuals, in the analysis of partial differential equations, Nash and Nirenberg influenced each other through their contributions and interactions. The consequences of their fruitful dialogue, which they initiated in the 1950s at the Courant Institute of Mathematical Sciences, are felt more strongly today than ever before.



A biography of John Forbes Nash, Jr.

Alexander Bellos

John F. Nash, Jr. is one of a handful of mathematicians known outside academia, due to the 2001 film about him, *A Beautiful Mind*, loosely based on Sylvia Nasar's bestselling biography of the same name. The Oscar-winning movie fictionalized Nash's path from brilliant Princeton student to being awarded the 1994 Nobel Prize for economics.

Inevitably, the Hollywood version of Nash's life story differed from the real one in many ways. In particular, the film focused on his early results in game theory, which have applications in economics, and omitted his research into geometry and partial differential equations, which the mathematical community regards as his most important and deepest work.

John Forbes Nash, Jr. was born in 1928 in Bluefield, West Virginia, a small, remote town in the Appalachians. His father was an electrical engineer at the local power company and his mother a schoolteacher. He entered the Carnegie Institute of Technology (now Carnegie Mellon University) in Pittsburgh with a full scholarship, originally studying for a major in chemical engineering, before switching to chemistry and finally changing again to mathematics.

At Carnegie, Nash took an elective course in economics, which gave him the idea for his first paper, *The Bargaining Problem*, which he wrote in his second term as a graduate student at Princeton University. This paper led to his interest in the new field of game theory – the mathematics of decision-making. Nash's Ph.D. thesis, *Non-Cooperative Games*, is one of the foundational texts of game theory. It introduced the concept of an equilibrium for non-cooperative games, the "Nash equilibrium", which has had a great impact in economics and the social sciences.

While at Princeton Nash also made his first breakthrough in pure mathematics. He described it as "a nice discovery relating to manifolds and real algebraic varieties." In essence the theorem shows that any manifold, a topological object like a surface, can be described by an algebraic variety, a geometric object defined by equations, in a much more concise way than had previously been thought possible. The result was already regarded by his peers as an important and remarkable work.

In 1951 Nash left Princeton to take up an instructorship at MIT. Here he became interested in the Riemann embedding problem, which asks whether it is possible to embed a manifold with specific rules about distance in some *n*-dimensional Euclidean space such that these rules are maintained. Nash provided two theorems that proved it was true: the first when smoothness was ignored and the second in a setting that maintained smoothness.

In order to prove his second embedding theorem, Nash needed to solve sets of partial differential equations that hitherto had been considered impossible to solve. He devised an iterative technique, which was then modified by Jürgen Moser, and is now known as the Nash–Moser theorem. The Abel Prize laureate Mikhail Gromov has said: "What [Nash] has done in geometry is, from my point of view, incomparably greater than what he has done in economics, by many orders of magnitude. It was an incredible change in attitude of how you think about manifolds. You can take them in your bare hands, and what you do may be much more powerful than what you can do by traditional means."

In the early 1950s Nash worked as a consultant for the RAND Corporation, a civilian think-tank funded by the military in Santa Monica, California. He spent a few summers there, where his work on game theory found applications in United States' military and diplomatic strategy.

Nash won one of the first Sloan Fellowships in 1956 and chose to take a year's sabbatical at the Institute of Advanced Study in Princeton. He based himself not in Princeton, but in New York, where he spent much of his time at Richard Courant's fledgling Institute for Applied Mathematics at NYU. It was here Nash met Louis Nirenberg, who suggested to him that he work on a major open problem in nonlinear theory concerning inequalities associated with elliptic partial differential equations. Within a few months Nash had proved the existence of these inequalities. Unknown to him, the Italian mathematician Ennio De Giorgi had already proved this, using a different method, and the result is known as the Nash-De Giorgi theorem.

Nash was not a specialist. He worked on his own, and relished tackling famous open problems, often coming up with completely new ways of thinking. In 2002 Louis Nirenberg said: "About twenty years ago somebody asked me, 'Were there any mathematicians you would consider as geniuses?' I said, 'I can think of one, and that's John Nash.'… He had a remarkable mind. He thought about things differently from other people."

In 1957 Nash married Alicia Larde, a physics major whom he met at MIT. In 1959 when Alicia was pregnant with their son, he began to suffer from delusions and extreme paranoia and as a result resigned from the MIT faculty. For the next three decades Nash was only able to do serious mathematical research in brief periods of lucidity. He improved gradually and by the 1990s his mental state had recovered.

The 1990s also saw him receive a number of honours for his professional work. As well as winning the prize in economic sciences in memory of Alfred Nobel in 1994, which he shared with John C. Harsanyi and Reinhard Selten, he was elected a member of the National Academy of Sciences in 1996, and in 1999 he won the American Mathematical Society's Steele Prize for Seminal Contribution to Research for his 1956 embedding theorem, sharing it with Michael G. Crandall.

A biography of Louis Nirenberg

Alexander Bellos

Louis Nirenberg has had one of the longest, most feted – and most sociable – careers in mathematics. In more than half a century of research he has transformed the field of partial differential equations, while his generosity, gift for exposition and modest charm have made him an inspirational figure to his many collaborators, students and colleagues.

Louis Nirenberg was born in Hamilton, Canada, in 1925 and grew up in Montreal, where his father was a Hebrew teacher. His first interest in mathematics came from his Hebrew tutor, who introduced him to mathematical puzzles. He studied mathematics and physics at McGill University, Montreal, avoiding the draft during World War II thanks to Canada's policy of exempting science students, and graduated in 1945.

The summer after graduating Nirenberg worked at the National Research Council of Canada on atomic bomb research. One of the physicists there was Ernest Courant, the elder son of New York University professor Richard Courant, who was building up NYU's mathematics department. Nirenberg asked Ernest's wife, who was a friend of his from Montreal, to ask her father-in-law for advice about where to do graduate studies in theoretical physics. Richard Courant responded that he should study *mathematics* at his department at NYU.

Nirenberg went for an interview in New York and was offered an assistantship. He got his masters in 1947, and embarked on a Ph.D. under James J. Stoker, who suggested to him an open problem in geometry that had been stated by Hermann Weyl three decades previously: can you embed isometrically a two-dimensional sphere with positive curvature into three Euclidean dimensions as a convex surface? In order to prove that you can, he reduced the problem to one about nonlinear partial differential equations. The PDEs in question were elliptic, a class of equations that have many applications in science. Nirenberg's subsequent work has been largely concerned with elliptic PDEs, and over the following decades he developed many important theorems about them.

Nirenberg never left mathematics, nor indeed NYU. Once he got his Ph.D. in 1949 he stayed on as a research assistant. He was a member of the faculty – known since 1965 as the Courant Institute of Mathematical Sciences – his entire career, becoming a full professor in 1957. Between 1970 and 1972 he was the Institute's director, and he retired in 1999. He still lives in Manhattan.

In the 1950s the Courant Institute was rapidly becoming one of the US's top research centres for applied mathematics, on a par with more established universities, although it only had a small number of staff. Nirenberg was one of its leading lights, and the mathematician who did the most work in providing a theoretical grounding for modern analysis of PDEs.

Nirenberg has always preferred to work in collaboration, with more than 90 per cent of his papers written jointly (none, however, with John F. Nash, Jr., whom Nirenberg got to know well during the academic year 1956–57). Important papers include results with his student August Newlander on complex structures in 1957, with Shmuel Agmon and Avron Douglis on regularity theory for elliptic equations in 1959, with Fritz John introducing the function space of functions with bounded mean oscillation in 1961, with David Kinderlehrer and Joel Spruck developing regularity theory for free boundary problems in 1978 and with Basilis Gidas and Wei Ming Ni about the symmetries of solutions of PDEs in 1979. A paper on solutions to the Navier–Stokes equations, co-authored with Luis A. Caffarelli and Robert V. Kohn, won the American Mathematical Society's 2014 Steele Prize for Seminal Contribution to Research.

As well as demonstrating vision and leadership, Nirenberg has shown remarkable energy and stamina, continuing to produce ground-breaking work in different areas of PDEs until his 70s. He is known not only for his technical mastery but also for his taste, instinctively knowing which are the problems worth spending time on. He has supervised more than forty Ph.D. students and is an excellent lecturer and writer.

Ever since he spent the academic year 1951–52 in Zürich, Switzerland, and Göttingen, Germany, Nirenberg has been a well-travelled and active member of the international mathematical community. On his first professional visit to Italy, in 1954 to attend a conference on PDEs, he immediately felt surrounded by friends. "That's the thing I try to get across to people who don't know anything about mathematics, what fun it is!" he has said. "One of the wonders of mathematics is you go somewhere in the world and you meet other mathematicians and it's like one big family. This large family is a wonderful joy." He was present at the first big US–Soviet joint maths conference in Novosibirsk in 1963, and in the 1970s was one of the first US mathematicians to visit China.

Nirenberg has gathered a significant number of prestigious accolades. He won the American Mathematical Society's Bôcher Memorial Prize in 1959. In 1969 he was elected to the National Academy of Sciences. He won the inaugural Crafoord Prize, awarded by the Royal Swedish Academy of Science and given in areas not covered by the Nobel Prizes, in 1982 (together with Vladimir Arnold). He received the Steele Prize for Lifetime Achievement from the American Mathematical Society in 1994, and he received the National Medal of Science in 1995, the highest honour in the US for contributions to science. In 2010 he was awarded the first Chern Medal for lifetime achievement by the International Mathematical Union and the Chern Medal Foundation.

Never change a given distance ...

Arne B. Sletsjøe

Neurons are not evenly distributed in the human body. Some parts of the body, like the hands, face and tongue are much more sensitive to sensations than other parts. The body has the highest density of neurons in those parts. A function that measures the density of neurons is an example of what mathematicians call a *metric*. Another example of a metric is the so-called Euclidean metric, named after the ancient Greek mathematician Euclid. The Euclidean metric measures ordinary distances between points and the area of any region of a surface. In a paper from 1916 Hermann Weyl asked the following question: *Is it always possible to realise an abstract metric on the 2-sphere of positive curvature by an isometric embedding in* $\mathbf{R}^3$*?* If you think of the neuron density metric as Weyl's abstract metric and the human body as the 2-sphere, then the weird body in figure 1 illustrates the positive answer to Weyl's question. The different sizes of the various body parts correspond to the neuron density.

The connection between Weyl's question and the work of Luis Nirenberg is emphasized in the citation of the Abel Prize: "Nirenberg, with his fundamental embedding theorems for the sphere S^2 in R^3, having prescribed Gauss curvature or Riemannian metric, solved the classical problems of Minkowski and Weyl."

A long time before spacecraft provided us with images of the earth, our forefathers concluded that our planet is round. They based this knowledge on observations done on the surface of the earth. By performing smart observations and correct measurements, they were able to conclude that the earth could not be flat. If you fix a point on a flat surface and you walk a circular path at a given distance R, the path should be $2\pi R$ long. But if you measure carefully on the earth's surface you will find that the perimeter is a little shorter. A theoretical computation then tells you that the earth's surface has positive curvature, i.e. locally it looks like a sphere.

The fact that it is possible to say anything about the curvature, merely by observations performed on the surface, was formulated by the great mathematician Carl Friedrich Gauss in 1827, in what is called Gauss' Theorema Egregium, *the remarkable theorem*. The theorem says that the Gaussian curvature of a surface can be determined entirely by measuring distances and angles on the surface itself, without further reference to how the surface is embedded in the three-dimensional space. Curvature is an *intrinsic* property of the surface, i.e. a property that belongs to the surface by its very nature. Consequently it has to be preserved by any isometric embedding.

In the first embedding theorem of John F. Nash, Jr., published in 1954, he proves that *any Riemannian manifold can be isometrically embedded in Euclidian space by a* C^1*-map*. The striking point of a curve version of this theorem is that any curve in the plane can be arbitrarily prolonged in a smooth way, without self-crossing and as close to the original curve as we want. The prolonged curve looks like the path of the front wheel of a bicycle climbing a steep hill, while the rear-wheel tracks out the original curve. By increasing the frequency of twists the cyclist can increase the difference between the length of the front-wheel path and the rear-wheel path. Unlike the surface case, curvature of a curve does not have to be preserved by an isometric embedding.

Whereas the one-dimensional version of Nash's theorem is rather intuitive, the two-dimensional version is more or less counter-intuitive, as the following illustration shows. Start with a piece of paper and turn it into a cylindrical shape. This is easy; the next step is the hard part: to turn the cylinder into a doughnut-shaped surface without stretching or tearing the paper. Intuitively this seems to be impossible. The outer circumference of the doughnut is much longer than the inner, but in the original cylinder they are of the same length. By Nash's theorem this is never the less possible, at least theoretically. Nash proved the theorem in 1954, but it was only in 2012 a multidisciplinary team in France, the HEVEA project, was able to image the process where the cylinder is bent into a doughnut, in an isometric way. The images in figure 2 illustrate the process; the paper is warped by an infinite sequence of waves, piling up to a doughnut surface in such a way that the original piece of paper is kept intact.

An **embedding** theorem in mathematics concerns itself with the extent to which it is possible to put one object into another, without "destroying" the objects.

An **isometric embedding** is an embedding where all distances between points are preserved. Distances are measured in the surface, so making a cylindrical shape out of a piece of paper is an isometric operation, whereas flattening out a sphere is a non-isometric operation.

Figure 1: The different size of the parts of the body reflects the density of neurons. Source: Natural History Museum, London

Figure 2: Images of an isometric embedding of a flat torus in R^3. Source: HEVEA Project/PNAS

About the Abel Prize

The Abel Prize is an international award for outstanding scientific work in the field of mathematics, including mathematical aspects of computer science, mathematical physics, probability, numerical analysis, scientific computing, statistics, and also applications of mathematics in the sciences. The Norwegian Academy of Science and Letters awards the Abel Prize based upon recommendations from the Abel Committee. The Prize is named after the exceptional Norwegian mathematician Niels Henrik Abel (1802–1829). According to the statutes of the Abel Prize, the objective is both to award the annual Abel Prize, and to contribute towards raising the status of mathematics in society and stimulating the interest of children and young people in mathematics. The prize carries a cash award of 6 million NOK (about 700,000 Euro or about 800,000 USD) and was first awarded in 2003. Among initiatives supported are the Abel Symposium, the International Mathematical Union's Commission for Developing Countries, the Abel Conference at the Institute for Mathematics and its Applications in Minnesota, and The Bernt Michael Holmboe Memorial Prize for excellence in teaching mathematics in Norway. In addition, national mathematical contests, and various other projects and activities are supported in order to stimulate interest in mathematics among children and youth.

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Call for nominations 2016:
The Norwegian Academy of Science and Letters hereby calls for nominations for the Abel Prize 2016, and invite you (or your society or institution) to nominate candidate(s). Nominations are confidential and a nomination should not be made known to the nominee.

Deadline for nominations for the Abel Prize 2016 is September 15, 2015. Please consult **www.abelprize.no** for more information

Yakov Sinai receives the Abel Prize from HRH Crown Prince Haakon in the University Aula, Oslo, May 2014.


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The Abel Prize Laureates

2014
Yakov G. Sinai

"for his fundamental contributions to dynamical systems, ergodic theory, and mathematical physics."

2013
Pierre Deligne

"for seminal contributions to algebraic geometry and for their transformative impact on number theory, representation theory, and related fields."

2012
Endre Szemerédi

"for his fundamental contributions to dis crete mathematics and theoretical computer science, and in recognition of the profound and lasting impact of these contributions on additive number theory and ergodic theory."

2008
John Griggs Thompson and Jacques Tits

"for their profound achievements in algebra and in particular for shaping modern group theory."

2007
Srinivasa S. R. Varadhan

"for his fundamental contributions to probability theory and in particular for creating a unified theory of large deviations."

2006
Lennart Carleson

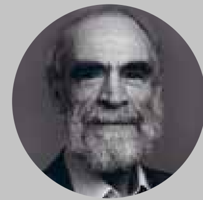
"for his profound and seminal contributions to harmonic analysis and the theory of smooth dynamical systems."

2011
John Milnor

"for pioneering discoveries in topology, geometry and algebra."

2010
John Torrence Tate

"for his vast and lasting impact on the theory of numbers."



2009
Mikhail Leonidovich Gromov

"for his revolutionary contributions to geometry."

2005
Peter D. Lax

"for his groundbreaking contributions to the theory and application of partial differential equations and to the computation of their solutions."

2004
Sir Michael Francis Atiyah and Isadore M. Singer

"for their discovery and proof of the index theorem, bringing together topology, geometry and analysis, and their outstand ing role in building new bridges between mathematics and theoretical physics."

2003
Jean-Pierre Serre

"for playing a key role in shaping the modern form of many parts of mathematics, including topology, algebraic geometry and number theory."

Programme
Abel Week 2015

May 18

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Holmboe Prize Award Ceremony
The Minister of Education and Research presents the Bernt Michael Holmboe Memorial Prize for teachers of mathematics at Oslo Cathedral School

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Wreath-laying at the Abel Monument
by the Abel Prize Laureates in the Palace Park

May 19

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Abel Prize Award Ceremony
His Majesty The King presents the Abel Prize in the University Aula, University of Oslo

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Reception and interview with the Abel Laureates
Science writer Vivienne Parry interviews the Abel Laureates at Det Norske Teatret

Abel Banquet at Akershus Castle in honor of the Abel Laureates
Hosted by the Norwegian Government (by invitation only from the Norwegian Government)

May 20

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The Abel Lectures
Laureate Lecture, Science Lecture, and other lectures in the field of the Laureates' work at Georg Sverdrups Hus, Aud. 1, University of Oslo

—

The Abel Party
at The Norwegian Academy of Science and Letters (by invitation only)

May 21

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Laureate Lectures and events for school children in Bergen
Programme at Festplassen, and Laureate lectures at the University of Bergen

ABEL PRIZE 2015

The Norwegian Academy of Science and Letters

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